Three-dimensional minimal CR submanifolds of the sphere $S^6(1)$ contained in a hyperplane

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Abstract

It is well known that the sphere $S^6(1)$ admits an almost complex structure J, constructed using the Cayley algebra, which is nearly Kaehler. Let M be a Riemannian submanifold of a manifold $\widetilde{M}$ with an almost complex structure J. It is called a CR submanifold if there exists a C^∞-differentiable holomorphic distribution $\mathcal{D}_1$ in the tangent bundle such that its orthogonal complement $\mathcal{D}_2$ in the tangent bundle is totally real. If the second fundamental form vanishes on $\mathcal{D}_i$, the submanifold is $\mathcal{D}_i$-geodesic. The first example of a 3-dimensional CR-submanifold was constructed by Sekigawa in [12]. This example was later generalised by Hashimoto and Mashimo in [11]. Note that both the original example as its generalisations are $\mathcal{D}_i$ geodesic.

Here, we investigate the class of the three-dimensional minimal CR submanifolds M of the nearly Kaehler 6-sphere $S^6(1)$ which are not linearly full. We show that this class coincides with the class of $\mathcal{D}_1$ and $\mathcal{D}_2$ geodesic CR submanifolds and we obtain a complete classification of such submanifolds.

1 Introduction

Considering $\mathbb{R}^7$ as the imaginary Cayley numbers, it is possible to introduce a vector cross product $\times$ on $\mathbb{R}^7$, which in its turn induces an almost complex structure J on the standard unit sphere $S^6(1)$ in $\mathbb{R}^7$ which is compatible with the standard metric. It was shown by Calabi and Gluck, see [4], that this structure, from a geometric viewpoint, is the best possible almost complex structure on $S^6(1)$. Details about this construction are recalled in Section 2.

With respect to the almost complex structure J, it is natural to study submanifolds for which J maps the tangent space into the tangent space (and hence also the normal space into the normal space) and those for which J maps the

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tangent into the normal space. The first class are called almost complex submanifolds and the second class of submanifolds mentioned are called totally real submanifolds.

One of the natural generalization of almost complex and totally real submanifolds are CR submanifolds and there are two different notions of this therm. By the first one, if the dimension of the holomorphic tangent space, the maximal J-invariant subspace $H_xM = JT_xM \cap T_xM, x \in M$ is independent on the choice of $x \in M$ then the submanifold M is called the Cauchy-Riemann submanifold, or briefly CR submanifold with the CR dimension being the constant complex dimension of H_xM. By the definition of Bejancu, see [2], a submanifold M is called a CR submanifold if there exists on M a differentiable holomorphic distribution $\mathcal{H}$ such that its orthogonal complement $\mathcal{H}^\perp \subset TM$ is a totally real distribution. It is clear that the CR submanifold by Bejancu's definition is also CR by the other definition. The converse is true for submanifolds of the maximal CR dimension $\dfrac{m-1}{2}$, where m is the dimension of the submanifold. A CR submanifold is called proper if it is neither totally real (i.e. $\mathcal{H}^\perp = TM$) nor almost complex (i.e. $\mathcal{H} = TM$).

CR submanifolds have been previously studied amongst others by K. Mashimo, H. Hashimoto and K. Sekigawa in [12] and [11]. In particular, in [11], the following one-parameter family of immersions of $S^2 \times \mathbb{R}$ was introduced:

$$\begin{aligned} F_\lambda((y_1, y_2, y_3), s) = {} & y_1(\cos se_1 + \sin se_5) \\ & + y_2(\cos \lambda se_2 + \sin \lambda se_5) + y_3(\cos(1+\lambda)se_3 - \sin(1+\lambda)se_7), \end{aligned}$$

where $y_1^2 + y_2^2 + y_3^2 = 1$ and $\{e_1, \ldots, e_7\}$ is a G_2 frame. Note that in [11] these examples were only defined for $\lambda \neq 0$ and $\lambda \neq -1$. However, it is easy to check that also for $\lambda \in \{0, -1\}$, the resulting immersion is a CR immersion with the same properties. Namely, it was shown that all of these examples satisfy:

1. the immersion is minimal
2. the immersion is contained in a totally geodesic hypersphere
3. the immersion is $\mathcal{D}_1$ totally geodesic
4. the immersion is $\mathcal{D}_2$ totally geodesic.

In Sections 4 and 5 we will show that for a CR submanifold the first and last two conditions in the above list are equivalent.

Another example of a minimal CR submanifold contained in a totally geodesic hypersphere of $S^6(1)$ is the CR submanifold which satisfy Chen's basic equality obtained in [7].

Also, the four-dimensional, minimal, CR submanifolds which are not linearly full were classified in [1]. In Section 5, we moreover investigate the three-dimensional minimal proper CR submanifolds which are not linearly full and obtain a complete classification. In particular we further generalise the class of examples obtained by Sekigawa, Hashimoto and Mashimo and show that this class can be charaterised by either of the following two conditions

1. the CR submanifold is minimal and contained in a totally geodesic hypersphere

2. the CR submanifold is $\mathcal{D}_1$ and $\mathcal{D}_2$ totally geodesic.

THEOREM 1.1. *Let M be a minimal three-dimensional CR submanifold of $S^6(1)$ which is not linearly full in $S^6(1)$. Then M is locally congruent to the immersion*

$$\begin{aligned} F(s,x_1,x_2) &= \cos x_1 \cos x_2(\cos(\mu_1)e_1 + \sin(\mu_1)e_5) + \sin x_1 \cos x_2(\cos(\mu_2)e_2 + \sin(\mu_2)e_6) \\ &\quad + \sin x_2(\cos(\mu_3)e_3 + \sin(\mu_3)e_7), \quad \mu_1+\mu_2+\mu_3 = 0, \quad \mu_1^2+\mu_2^2+\mu_3^2 \neq 0, \end{aligned}$$

where $e_1,\ldots,e_7$ is a standard G_2 basis of the space $\mathbb{R}^7$.

2 Preliminaries

We give a brief exposition of how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from the Cayley multiplication. For further details about the Cayley numbers and their automorphism group G_2, we refer the reader to [14] and [10].

The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product $\times$ on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula

$$u \times v = \frac{1}{2}(uv - vu), \tag{1}$$

while the standard inner product on $\mathbb{R}^7$ is given by

$$(u,v) = -\frac{1}{2}(uv + vu). \tag{2}$$

It is now elementary [10] to show that

$$u \times (v \times w) + (u \times v) \times w = 2(u,w)v - (u,v)w - (w,v)u, \tag{3}$$

and that the triple scalar product $(u \times v, w)$ is skew symmetric in u,v,w. From this it also follows that

$$< u \times v, u \times w >=< u,u >< v,w > - < u,v >< u,w > \tag{4}$$

The Cayley multiplication on $\mathcal{O}$ is given in terms of the vector cross product and the inner product by

$$(r+u)(s+v) = rs-(u,v)+rv+su+(u\times v), \quad r,s \in Re(\mathcal{O}), u,v \in Im(\mathcal{O}). \tag{5}$$

In view of (1), (2) and (5), it is clear that the group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\mathbb{R}^7$ preserving the vector cross product.

An ordered basis $e_1, ..., e_7$ is said to be a G_2-frame if

$$e_3 = e_1 \times e_2, \quad e_5 = e_1 \times e_4, \quad e_6 = e_2 \times e_4, \quad e_7 = e_3 \times e_4. \tag{6}$$

For example, the standard basis $e_1, ..., e_7$ of $\mathbb{R}^7$ is a G_2-frame. Two G_2-frames are related by a unique element of G_2. Moreover, if e_1, e_2, e_4 are mutually orthogonal unit vectors with e_4 orthogonal to $e_1 \times e_2$, then e_1, e_2, e_4 determine a unique G_2-frame $e_1, ..., e_7$ and $(\mathbb{R}^7, \times)$ is generated by e_1, e_2, e_4 subject to the relations :

$$e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik} e_j - \delta_{ij} e_k - \delta_{jk} e_i. \tag{7}$$

Therefore, for any G_2-frame, we have the following very useful multiplication table [14] :

x	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

The standard nearly Kähler structure on $S^6(1)$ is then obtained as follows :

$$Ju = x \times u, \quad u \in T_x S^6(1), \quad x \in S^6(1).$$

It is clear that J is an orthogonal almost complex structure on $S^6(1)$. In fact J is a nearly Kähler structure in the sense that the $(2,1)$-tensor field G on $S^6(1)$ defined by

$$G(X,Y) = (\tilde{\nabla}_X J)Y,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$ is skew-symmetric. If we denote by $\langle\ ,\ \rangle$ the metric of the space $\mathbb{R}^7$, a straightforward computation also shows that

$$G(X,Y) = X \times Y - \langle x \times X, Y\rangle x, \quad X, Y \in T_x S^6(1).$$

Let M be a Riemannian submanifold of $\widetilde{M}$. If we denote by $\langle\ ,\ \rangle$, $\overline{D}$ and $\widetilde{D}$ metric and Levi Civita connections on M and $\widetilde{M}$, respectively, and by $D^{\perp}$ the corresponding normal connection of the immersion $M \to \widetilde{M}$ then the formulas of Gauss and Weingarten are given by

$$\widetilde{D}_X Y = \overline{D}_X Y + h(X,Y), \tag{8}$$

$$\widetilde{D}_X \xi = -A_\xi X + D^{\perp}_X \xi, \tag{9}$$

where X and Y are vector fields on M and ξ is a normal vector field on M, and h and A are the second fundamental form and the shape operator, respectively. The second fundamental form and the shape operator are related by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle. \tag{10}$$

Let us denote by $\nabla, \widetilde{\nabla}$ and D the Levi-Civita connections on $M, S^6(1)$ and R^7, respectively. Let h and $\widetilde{h}$ be the second fundamental forms corresponding to the immersions $M \to S^6(1)$ and $S^6(1) \to R^7$, respectively. Let p be the position vector field of the immersion of M into R^7. Then the following equations hold

$$\widetilde{h}(X,Y) = -\langle X,Y\rangle p, \tag{11}$$

$$D_X p = X, \tag{12}$$

where $X, Y \in TM$. Considering (8), (9) and (11) we get for $X, Y \in TM$ and $\xi \in T^\perp M, \xi \in TS^6(1)$

$$\begin{aligned} D_X Y &= \widetilde{\nabla}_X Y + \widetilde{h}(X,Y) = \nabla_X Y + h(X,Y) - \langle X,Y\rangle p, && (13)\\ D_X \xi &= \widetilde{\nabla}_X \xi + \widetilde{h}(X,\xi) = \widetilde{\nabla}_X \xi - \langle X,\xi\rangle p = -A_\xi X + \nabla^\perp_X \xi, && (14)\end{aligned}$$

where $\nabla^\perp$ denotes the normal connection corresponding to the immersion of M into $S^6(1)$. Also, we can denote

$$(\nabla h)(X,Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{15}$$

for $X, Y, Z \in T(M)$. Then Gauss, Codazzi and Ricci equations state that

$$\begin{aligned} R(X,Y,Z,W) &= \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \\ &\quad + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle, && (16)\\ (\nabla h)(X,Y,Z) &= (\nabla h)(Y,X,Z), && (17)\\ \langle R^\perp(X,Y)\xi,\mu\rangle &= \langle [A_\xi, A_\mu]X, Y\rangle, && (18)\end{aligned}$$

Also the following lemma holds

Lemma 2.1. *$D_X(Y \times Z) = D_X Y \times Z + Y \times D_X Z$.*

3 Three-dimensional CR submanifolds of the sphere $S^6(1)$

From now on we consider M to be a three-dimensional orientable CR submanifold of the sphere $S^6(1)$. Then, there exist the following local orthonormal vector fields: the position vector field p, E_1 and $E_2 = JE_1$ which span the almost complex distribution, E_3 which spans the totally real distribution, and the normal vector fields $E_4 = JE_3, E_5 = E_1 \times E_3$ and $E_6 = E_2 \times E_3$.

Note, that by assuming that E_1, E_2 and E_3 are positively oriented, we have that the choice of E_3 is unique. Nevertheless, we still have the following freedom:

$$\begin{aligned}
&\tilde{E}_1 = \cos\theta E_1 + \sin\theta E_2, && \tilde{E}_2 = J\tilde{E}_1 = -\sin\theta E_2 + \cos\theta E_1,\\
&\tilde{E}_3 = E_3, && \tilde{E}_4 = E_4,\\
&\tilde{E}_5 = (\cos\theta E_5 + \sin\theta E_6), && \tilde{E}_6 = (-\sin\theta E_5 + \cos\theta E_6).
\end{aligned}$$

As M is a CR submanifold we already have that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are respectively the almost complex and the totally real distribution.
Using the standard symmetries for a connection and for the second fundamental form, we find that

$$\begin{aligned}
&D_{E_1}E_1 = -p + a_1E_2 + a_2E_3 + \alpha_1E_4 + \alpha_2E_5 + \alpha_3E_6, && D_{E_1}E_2 = -a_1E_1 + a_3E_3 + \beta_1E_4 + \beta_2E_5 + \beta_3E_6,\\
&D_{E_1}E_3 = -a_2E_1 - a_3E_2 + \gamma_1E_4 + \gamma_2E_5 + \gamma_3E_6, && D_{E_1}E_4 = -\alpha_1E_1 - \beta_1E_2 - \gamma_1E_3 + g_1E_5 + g_2E_6,\\
&D_{E_1}E_5 = -\alpha_2E_1 - \beta_2E_2 - \gamma_2E_3 - g_1E_4 + g_3E_6, && D_{E_1}E_6 = -\alpha_3E_1 - \beta_3E_2 - \gamma_3E_3 - g_2E_4 - g_3E_5,\\
&D_{E_2}E_1 = b_1E_2 + b_2E_3 + \beta_1E_4 + \beta_2E_5 + \beta_3E_6, && D_{E_2}E_2 = -p - b_1E_1 + b_3E_3 + \delta_1E_4 + \delta_2E_5 + \delta_3E_6,\\
&D_{E_2}E_3 = -b_2E_1 - b_3E_2 + \mu_1E_4 + \mu_2E_5 + \mu_3E_6, && D_{E_2}E_4 = -\beta_1E_1 - \delta_1E_2 - \mu_1E_3 + h_1E_5 + h_2E_6,\\
&D_{E_2}E_5 = -\beta_2E_1 - \delta_2E_2 - \mu_2E_3 - h_1E_4 + h_3E_6, && D_{E_2}E_6 = -\beta_3E_1 - \delta_3E_2 - \mu_3E_3 - h_2E_4 - h_3E_5,\\
&D_{E_3}E_1 = c_1E_2 + c_2E_3 + \gamma_1E_4 + \gamma_2E_5 + \gamma_3E_6, && D_{E_3}E_2 = -c_1E_1 + c_3E_3 + \mu_1E_4 + \mu_2E_5 + \mu_3E_6,\\
&D_{E_3}E_3 = -p - c_2E_1 - c_3E_2 + \nu_1E_4 + \nu_2E_5 + \nu_3E_6, && D_{E_3}E_4 = -\gamma_1E_1 - \mu_1E_2 - \nu_1E_3 + k_1E_5 + k_2E_6,\\
&D_{E_3}E_5 = -\gamma_2E_1 - \mu_2E_2 - \nu_2E_3 - k_1E_4 + k_3E_6, && D_{E_3}E_6 = -\gamma_3E_1 - \mu_3E_2 - \nu_3E_3 - k_2E_4 - k_3E_5,
\end{aligned}$$

for some local functions.
Straightforward computation, taking in Lemma 1, $X \in \{E_1, E_2, E_3\}$ and $Y, Z \in \{p, E_1, \ldots, E_6\}$ we get the following lemma.

LEMMA 3.1. *For the previously defined coefficient the following equations hold*

$$\begin{aligned}
&g_2 = -\gamma_2, \quad g_1 = 1 + \gamma_3, \quad \alpha_1 = -a_3, \quad \beta_1 = a_2, \quad h_2 = 1 - \mu_2, \quad h_1 = \mu_3,\\
&\delta_1 = b_2, \quad b_3 = -a_2, \quad k_1 = \nu_3, \quad k_2 = -\nu_2, \quad \mu_1 = c_2, \quad \gamma_1 = -c_3,\\
&\alpha_3 = \beta_2, \quad \alpha_2 = -\beta_3, \quad \delta_2 = \beta_3, \quad \delta_3 = -\beta_2, \quad \mu_2 = \gamma_3 - 1, \quad \mu_3 = -\gamma_2,\\
&g_3 = a_1 - c_3, \quad h_3 = b_1 + c_2, \quad k_3 = c_1 + \nu_1.
\end{aligned}$$

4 $\mathcal{D}_1$ and $\mathcal{D}_2$-geodesic CR submanifolds

Since there are no three-dimensional, proper CR, totally geodesic submanifolds of the sphere S^6, it is natural to investigate submanifolds that in some sense approach this quality. Namely, we investigate three-dimensional CR submanifolds for which corresponding second fundamental form vanishes on $\mathcal{D}_1$ and $\mathcal{D}_2$. Such submanifolds are called, respectively, $\mathcal{D}_1$-geodesic and $\mathcal{D}_2$-geodesic. If the submanifold is both $\mathcal{D}_1$ and $\mathcal{D}_2$-geodesic it is trivially minimal. One example of such submanifold was given in [12].
In this section we assume that M is both $\mathcal{D}_1$ and $\mathcal{D}_2$-geodesic. It follows

$$a_3 = \beta_3 = \beta_2 = a_2 = b_2 = \nu_1 = \nu_2 = \nu_3 = 0.$$

This also immediately implies that M is a minimal submanifold.

Note that the vector field E_3 is uniquely determined up to a sign. That also means that the vector field $D_{E_3}E_3$ is independent of the choice of the basis. Therefore, we can choose vector field E_1 such that $D_{E_3}E_3$ is orthogonal to E_1, meaning $c_2 = 0$. Can $\nabla_{E_3}E_3$ be totally real? Suppose it is possible, meaning that $c_3 = 0$. Than from $R(E_2, E_3, E_1, E_3) = 0$ we obtain $E_2(c_2) = b_1c_3 - 2c_2c_3 - \gamma_2 = 0$ which implies $\gamma_2 = 0$. From $R(E_1, E_3, E_1, E_3) = 0$ we get $E_1(c_2) = \gamma_3^2 - 1 = 0$ and further $\gamma_3^2 = 1$. From $R(E_1, E_2, E_3, E_5) = E_2(c_3) - (\gamma_3 - 2)\gamma_3 = 0$ we obtain a contradiction with $\gamma_3^2 = 1$.

Gauss and Codazzi equations give now, some new relations among the coefficients.

LEMMA 4.1.

$$\gamma_2 = b_1c_3, \quad 0 = -1 + a_1c_3 + c_3^2 + \gamma_2^2 + \gamma_3^2, \quad c_1 = 0, \quad E_2(\gamma_3) = -2c_3(-1 + b_1^2 + \gamma_3),$$
$$E_3(b_1) = 0, \quad E_1(c_3) = -b_1c_3, \quad E_1(\gamma_3) = 2b_1(-1 + (1 + b_1^2)c_3^2 + \gamma_3^2), \quad E_3(c_3) = 0,$$
$$E_3(\gamma_3) = 0, \quad E_2(c_3) = (-1 + b_1^2)c_3^2 + (-2 + \gamma_3)\gamma_3, \quad E_2(b_1) = -\frac{b_1(1 + c_3^2 + b_1^2c_3^2 - 4\gamma_3 + \gamma_3^2)}{c_3},$$
$$E_1(b_1) = -\frac{(-1 + \gamma_3)(-1 + 2c_3^2 + 2b_1^2c_3^2 + \gamma_3 + 2\gamma_3^2)}{c_3^2}.$$

PROOF. Gauss equation for $R(E_2, E_3, E_1, E_3)$ yields $\gamma_2 = b_1c_3$. Further from $R(E_1, E_3, E_1, E_3) = 0$ we obtain

$$0 = -1 + a_1c_3 + c_3^2 + \gamma_2^2 + \gamma_3^2.$$

Also, $R(E_2, E_3, E_3, E_4) = c_1c_3 = 0$ gives $c_1 = 0$. Directly from the Gauss equation for $R(E_1, E_3, E_1, E_2)$ we now get $E_3(b_1) = 0$, while $R(E_2, E_3, E_6, E_1) = 0$ gives $E_2(\gamma_3) = -2c_3(-1+b_1^2+\gamma_3)$. Similarly, Codazzi equations for $R(E_1, E_3, E_1, E_4) = 0$, $R(E_1, E_3, E_1, E_6) = 0$, $R(E_1, E_3, E_3, E_4) = 0$, $R(E_1, E_3, E_3, E_6) = 0$, $R(E_2, E_3, E_1, E_4) = 0$ and $R(E_2, E_3, E_1, E_5) = 0$, respectively give

$$E_1(c_3) = -b_1c_3, \qquad E_1(\gamma_3) = 2b_1(-1 + (1 + b_1^2)c_3^2 + \gamma_3^2),$$
$$E_3(c_3) = 0, \qquad E_3(\gamma_3) = 0,$$
$$E_2(c_3) = (-1 + b_1^2)c_3^2 + (-2 + \gamma_3)\gamma_3, \quad E_2(b_1) = -\frac{b_1(1 + c_3^2 + b_1^2c_3^2 - 4\gamma_3 + \gamma_3^2)}{c_3}.$$

Finally, from Gauss equation for $R(E_1, E_2, E_1, E_2)$ we obtain

$$E_1(b_1) = -\frac{(-1 + \gamma_3)(-1 + 2c_3^2 + 2b_1^2c_3^2 + \gamma_3 + 2\gamma_3^2)}{c_3^2}.$$

Straightforward computation shows that other Gauss, Codazzi and Ricci equations don't yield any new relations. □

Also, these relations satisfy integrability conditions.

LEMMA 4.2. *Let M be three-dimensional connected CR submanifold of the sphere S^6 and $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$ where $\mathcal{D}_1$ and $\mathcal{D}_2$ are corresponding almost complex and totally real distribution. Let $h(\mathcal{D}_1, \mathcal{D}_1) = h(\mathcal{D}_2, \mathcal{D}_2) = 0$. Then M is minimal and contained in a totally geodesic hypersphere.*

PROOF. As $\mathcal{D}_1$ and $\mathcal{D}_2$ are totally geodesic it immediately follows that m is mininal and that the first normal space of the submanifold M is spanned by vector fields

$$n_1 = h(E_1, E_3) = -c_3E_4 + b_1c_3E_5 + \gamma_3E_6,$$
$$n_2 = h(E_2, E_3) = (-1 + \gamma_3)E_5 - b_1c_3E_6.$$

Then straightforward computation shows that

$$\nabla^{\perp}_{E_1}n_1 = -\frac{-1 + (1 + b_1^2)c_3^2 + \gamma_3^2}{c_3}n_2, \;\; \nabla^{\perp}_{E_2}n_1 = -c_3n_1 + b_1n_2, \;\; \nabla^{\perp}_{E_3}n_1 = 0.$$

Similarly,

$$\nabla^{\perp}_{E_1}n_2 = -\frac{1 - b_1^2c_3^2 - \gamma_3^2}{c_3}n_1, \;\; \nabla^{\perp}_{E_2}n_2 = -2c_3n_2 - b_1n_1, \;\; \nabla^{\perp}_{E_3}n_2 = 0.$$

We conclude that the first normal space of the submanifold is invariant under parallel translations with respect to the connection in the normal bundle and by the Erbacher's theorem it follows that the submanifold M is not linearly full. □

In the Section 5 we will conclude that the converse also holds, i.e. a minimal CR submanifold contained in a totally geodesic S^5 is $\mathcal{D}_1$ and $\mathcal{D}_2$ totally geodesic and therefore satisfies the conditions of Lemma 4.2.

5 The proof of the Main Theorem

From now on we will assume that M minimal, three-dimensional CR submanifold contained in a totally geodesic S^5 in $S^6(1)$. As a totally geodesic hypersphere is obtained by taking the intersection of $S^6(1)$ with a hyperplane through the origin, it follows that there exists a constant unit length vector field V, namely the unit normal to that plane, which is normal to the submanifold M and tangent to the sphere $S^6(1)$.
Therefore we can write

$$V = \rho E_4 + \tau E_5 + \sigma E_6.$$

And using the rotation freedom in our basis we can moreover assume that $\tau = 0$.
As V is unit length, we also have that $\rho^2 + \sigma^2 = 1$.
Moreover, as M is also a minimal submanifold, we have

$$-a_3 + b_2 + \nu_1 = \nu_2 = \nu_3 = 0.$$

Using the fact that V is constant, we have the following lemma which gives additional relations between the unknown local functions.

LEMMA 5.1. *Let ρ and σ be previously defined coefficients. Then we have*

$$\nu_1 = 0, \quad b_2 = a_3, \quad c_1 = 0, \quad \beta_2 = a_3\frac{\rho}{\sigma}, \quad \beta_3 = -a_2\frac{\rho}{\sigma}, \quad c_3 = \frac{\sigma(-\rho + a_1\sigma)}{\rho^2 + \sigma^2}, \quad \gamma_2 = -\frac{b_1\rho\sigma}{\rho^2 + \sigma^2},$$
$$c_2 = -\frac{b_1\sigma^2}{\rho^2 + \sigma^2}, \quad \gamma_3 = \frac{\rho(-\rho + a_1\sigma)}{\rho^2 + \sigma^2}, \quad E_1(\rho) = \frac{b_1\rho\sigma^2}{\rho^2 + \sigma^2}, \quad E_1(\sigma) = -\frac{b_1\rho^2\sigma}{\rho^2 + \sigma^2},$$
$$E_2(\rho) = \sigma(2 - \frac{\rho(-\rho + a_1\sigma)}{\rho^2 + \sigma^2}), \quad E_2(\sigma) = \rho(-2 + \frac{\rho(-\rho + a_1\sigma)}{\rho^2 + \sigma^2}), \quad E_3(\rho) = 0, \quad E_3(\sigma) = 0.$$

PROOF. Since the vector field V is constant it follows $D_X V = 0$ for any vector field X. Then

$$
\begin{aligned}
D_{E_1}V &= (a_3\rho - \beta_2\sigma)E_1 + (-a_2\rho - \beta_3\sigma)E_2 + (c_3\rho - \gamma_3\sigma)E_3 \\
&\quad + (\gamma_2\sigma + E_1(\rho))E_4 + ((1+\gamma_3)\rho + (-a_1 + c_3)\sigma)E_5 + (-\gamma_2\rho + E_1(\sigma))E_6, \\
D_{E_2}V &= (-a_2\rho - \beta_3\sigma)E_1 + (-b_2\rho + \beta_2\sigma)E_2 + (-c_2\rho + \gamma_2\sigma)E_3 \\
&\quad + (-2+\gamma_3)\sigma + E_2(\rho))E_4 + (-\gamma_2\rho + (-b_1 - c_2)\sigma))E_5 + ((2-\gamma_3)\rho + E_2(\sigma))E_6, \\
D_{E_3}V &= (c_3\rho - \gamma_3\sigma)E_1 + (-c_2\rho + \gamma_2\sigma)E_2 + (-a_3 + b_2)\rho E_3 \\
&\quad + E_3(\rho)E_4 + (-a_3 + b_2 - c_1)\sigma E_5 + E_3(\sigma)E_6.
\end{aligned}
$$

Suppose $\rho = 0$. Then $\sigma \neq 0$ since V is nonzero. From $\langle D_{E_1}V, E_3\rangle = 0$ we get $\gamma_3 = 0$, and from $\langle D_{E_2}V, E_4\rangle = 0$ we get a contradiction $\gamma_3 = 2$. Therefore $\rho \neq 0$ and considering $\langle D_{E_3}V, E_3\rangle = 0$ we get $a_3 = b_2$ and $\nu_1 = 0$. Similarly $\sigma \neq 0$, since otherwise $\langle D_{E_1}V, E_5\rangle = 0$ contradicts $\langle D_{E_2}V, E_6\rangle = 0$. Then from $\langle D_{E_3}V, E_5\rangle = 0$ we get $c_1 = 0$. Other equalities follow directly. □

Note that from the proof of the previous lemma it follows that both σ and ρ cannot vanish on an open subset. We therefore restrict to the open dense subset of M on which there are non vanishing. Hence we can write $\sigma = \rho t$, where t is a local non zero function. As V is unit length, we also deduce that $\rho^2(t^2+1) = 1$. From the previous proof it now follows that

$$
E_1(t) = -tb_1, \quad E_2(t) = -3 + a_1 t - 2t^2, \quad E_3(t) = 0. \tag{19}
$$

Now we will use the Gauss and Codazzi equations to obtain further relations between the coefficients.

LEMMA 5.2. *Let a_1, b_1 and t be the previously defined coefficients. Then we have*

$$
\begin{aligned}
&a_2 = 0, \quad a_3 = 0, \quad E_1(a_1) = 3a_1b_1, \quad E_1(b_1) = 3a_1\frac{1}{t} + 1 - 2a_1^2 + b_1^2, \\
&E_2(a_1) = 2 - a_1^2 + 2b_1^2 + 3a_1\frac{1}{t}, \quad E_2(b_1) = 6b_1\frac{1}{t} - 3a_1b_1, \quad E_3(a_1) = 0, \quad E_3(b_1) = 0.
\end{aligned}
$$

PROOF. The Gauss equations for $R(E_1, E_3, E_1, E_2)$ and $R(E_2, E_3, E_1, E_2)$, the Ricci equation for $R(E_1, E_3, E_6, E_4)$ and the Codazzi equation for $R(E_2, E_3)E_3$ directly imply that the following expressions, respectively, equal zero:

$$
\begin{aligned}
y_1 &= 3a_1a_2 + 3a_3b_1 - \frac{3a_2}{t} - E_3(a_1), \\
z_1 &= 3a_1a_3 - 3a_2b_1 - \frac{3a_3}{t} - E_3(b_1), \\
y_4 &= -a_2b_1\rho^2 t + a_3(-3\rho^2 + a_1\rho^2 t - 2\rho^2 t^2) + \rho^2 t E_3(b_1), \\
z_2 &= 3a_2\rho^2 - a_1a_2\rho^2 t - a_3b_1\rho^2 t + 2a_2\rho^2 t^2 - \rho^2 t E_3(a_1).
\end{aligned}
$$

Let us denote $x = -4b_1\rho^2 t$ and $y = 3\rho^2 - 2a_1\rho^2 t + \rho^2 t^2$. Then the equations

$$
\begin{aligned}
z_2 - \rho^2 t y_1 &= 0, \\
y_4 + \rho^2 t z_1 &= 0
\end{aligned}
$$

simplify to the system $a_3x + a_2y = 0$, $a_2x - a_3y = 0$. Suppose first that $a_2^2 + a_3^2 \neq 0$. Then $x = 0$, $y = 0$, i.e. $b_1 = 0$ and $a_1 = \frac{3+t^2}{2t}$. Also $a_3y + y_4 = 0$ implies $-a_1a_3 = a_3t$ and $z_1 = 0$ implies $a_1a_3t = a_3$ which reduces to $a_3 = 0$. Now, $E_3(t) = 0$ and

$$0 = y_1 = \frac{9+3t^2}{2t}a_2 - \frac{3}{t}a_2 - E_3(\frac{3+t^2}{2t}) = \frac{3+3t^2}{t}a_2$$

directly implies $a_2 = 0$, which is a contradiction. Hence we must have $a_2 = a_3 = 0$. Consequently also $E_3(a_1) = E_3(b_1) = 0$. The other equalities follow in a similar way. $\square$

Summarizing the previous lemmas, we have the following theorem.

Theorem 5.3. *Let M be a minimal three-dimensional CR submanifold of $S^6(1)$ which is not linearly full in $S^6(1)$. Then, restricting to an open dense subset, there exist tangent vector fields E_1, E_2, E_3 to M, normal vector fields E_4, E_5, E_6 and local functions a_1, b_1 and t such that the induced connection is given by*

$$\begin{aligned}
&\nabla_{E_1}E_1 = a_1E_2, && \nabla_{E_1}E_2 = -a_1E_1, && \nabla_{E_1}E_3 = 0,\\
&\nabla_{E_2}E_1 = b_1E_2, && \nabla_{E_2}E_2 = -b_1E_1, && \nabla_{E_2}E_3 = 0,\\
&\nabla_{E_3}E_1 = -\frac{b_1t^2}{1+t^2}E_3, && \nabla_{E_3}E_2 = \frac{t(a_1t-1)}{1+t^2}E_3, && \nabla_{E_3}E_3 = \frac{b_1t^2}{1+t^2}E_1 + \frac{t-a_1t^2}{1+t^2}E_2
\end{aligned} \tag{20}$$

and the second fundamental form is given by

$$\begin{aligned}
&h(E_1,E_1) = 0,\ \ h(E_1,E_2) = 0,\ \ h(E_1,E_3) = \frac{t-a_1t^2}{1+t^2}E_4 - \frac{b_1t}{1+t^2}E_5 + \frac{-1+a_1t}{1+t^2}E_6,\\
&h(E_2,E_2) = 0,\ \ h(E_2,E_3) = -\frac{b_1t^2}{1+t^2}E_4 + \frac{-2+a_1t-t^2}{1+t^2}E_5 + \frac{b_1t}{1+t^2}E_6,\ \ h(E_3,E_3) = 0.
\end{aligned}$$

Moreover, the functions a_1, b_1, t satisfy the following system of differential equations:

$$\begin{aligned}
&E_1(a_1) = 3a_1b_1, && E_2(a_1) = 2 - a_1^2 + 2b_1^2 + 3a_1\tfrac{1}{t}, && E_3(a_1) = 0,\\
&E_1(b_1) = 3\tfrac{a_1}{t} + 1 - 2a_1^2 + b_1^2, && E_2(b_1) = 6\tfrac{b_1}{t} - 3a_1b_1, && E_3(b_1) = 0,\\
&E_1(t) = -tb_1, && E_2(t) = -3 + a_1t - 2t^2, && E_3(t) = 0.
\end{aligned}$$

Using the previous expressions for the connection coefficients, we conclude that

$$[E_1,E_2] = -a_1E_1 - b_1E_2, \qquad [E_1,E_3] = \frac{b_1t^2}{1+t^2}E_3, \qquad [E_2,E_3] = -\frac{t(-1+a_1t)}{1+t^2}E_3.$$

In particular, we remark that these vector fields do not define local coordinates.

Example 5.4. Let us recall the basic inequality, discovered by B. Y. Chen in [6] for arbitrary n–dimensional submanifold $\widetilde{M}$ of a real space form of a constant sectional curvature c. This inequality relates a basic intrinsic invariant $\delta_{\widetilde{M}}$, with the length of the mean curvature vector H. Namely, if we denote by $\inf K$ at the point p infimum of the sectional curvature $K(\pi)$ of planes π in $T_p\widetilde{M}$ and scalar curvature by $\tau = \sum_{i<j} K(e_i \wedge e_j)$ where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space $T_p\widetilde{M}$ then $\delta_{\widetilde{M}}$ is given by $\delta_{\widetilde{M}}(p) = \tau(p) - \inf K(p)$, and it satisfies

$$\delta_{\widetilde{M}} \leq \frac{n^2(n-2)}{2(n-1)}H^2 + \frac{1}{2}(n+1)(n-2)c.$$

If submanifold satisfies the equality case of this inequality there exists a canonical distribution $\mathcal{D}(p) = \{X \in T_pM | (n-1)h(X,Y) = n\langle X, Y\rangle H, \forall Y \in T_p\widetilde{M}\}$. We recall here the following result from [6] which is here formulated for the three-dimensional submanifolds of $S^6(1)$.

LEMMA 5.5. *Let $\widetilde{M}$ be a three-dimensional submanifold of the sphere $S^6(1)$. Then $\delta_{\widetilde{M}} \leq \frac{9}{4}H^2 + 2$ and equality holds at a point p if and only if the dimension of $\mathcal{D} = \{X \in T_p\widetilde{M} | (n-1)h(X,Y) = n\langle X, Y\rangle H, \forall Y \in T_p\widetilde{M}\}$ is greater or equal to one.*

Notice now, that the space of the second fundamental form for submanifold M is one-dimensional if and only if $x = 2 + a_1^2t^2 + (1 + b_1^2)t^2 - a_1t(3 + t^2)$ vanishes. In other words, in this case, a non-zero vector field V defined by $V = b_1tE_1 - (-1 + a_1t)E_2$ satisfies $h(V, E_i) = 0, i \in \{1, 2, 3\}$. Since M is minimal, i.e. $H = 0$ it follows that corresponding distribution $\mathcal{D}$ is at least one-dimensional and M satisfies Chen's equality. We now refer to [7] and conclude that M is locally congruent with the immersion

$$f(s, x_1, x_2) = \cos x_1 \cos x_2(\cos se_1 - \sin se_5) + \sin x_2 e_2 + \sin x_1 \cos x_2(\cos se_3 + \sin se_7),$$

which satisfies the condition of the main theorem. It will be useful to notice that the function $x = 2 + a_1^2t^2 + (1 + b_1^2)t^2 - a_1t(3 + t^2)$ satisfies the following system of differential equations:

$$E_1(x) = 0, \quad E_2(x) = -6tx, \quad E_3(x) = 0. \tag{21}$$

Now, we return to general case. Note that $h(X, Y) = 0, X, Y \in \mathcal{D}_1$ and $h(E_3, E_3) = 0$, which means that for the minimal submanifolds the converse of the Lemma 4.2 holds. Also, the distribution $\mathcal{D}_1$ is involutive with totally geodesic leaves both in M and in S^6, so they are totally geodesic and almost complex spheres S^2.

Note that we are working on an open dense subset U such that the constant normal vector field V has components in both the spaces $J\mathcal{D}_2$ and $\mathcal{D}_2 \times \mathcal{D}_1$.

For a given point p of the open dense subset of the submanifold M constructed previously (such that the function t is a non vanishing function) we can assume that the coordinate system of the R^7 is such that p has coordinates $e_1 = (1, 0, \ldots, 0)$, $E_1(p) = (0, 1, 0, 0, 0, 0, 0)$ and such that the normal vector to the totally geodesic $S^5(1)$ containing M^3 is given by e_4. Note that we still have the freedom to choose the sign of e_4 appropriately. We parametrize the corresponding leaf S_0^2 by $(\cos x_1 \cos x_2, \cos x_1 \sin x_2, \sin x_1, 0, 0, 0, 0), x_1 \in (-\pi/2, \pi/2), x_2 \in (-\pi, \pi)$. Let us denote by $\gamma : I \to S^6$ the integral curve of the vector field E_3 with the initial condition that $\gamma(0) = p$. For each point $\gamma(s_0)$

there is a unique G_2-isometry, denoted by $A(s_0)$ of the sphere S^6 mapping the S_0^2 into the corresponding leaf through $\gamma(s_0)$ defined by the conditions

$$\begin{aligned} A(s_0)(e_1) &= \gamma(s_0) \\ A(s_0)(e_4) &= e_4 \\ A(s_0)(e_2) &= E_1(\gamma(s_0)). \end{aligned}$$

Note that from the above conditions it follows that A depends differentiably on the parameter s of the integral curve. Therefore, the manifold M is locally given by

$$F(x_1, x_2, s) = A(s)(\cos x_1 \cos x_2, \sin x_1 \cos x_2, \sin x_2, 0, 0, 0, 0)^t.$$

Let us denote by $A_1(s), A_2(s), A_3(s)$ the first three columns of $A(s)$. Then, $F(x_1, x_2, s) = \cos x_1 \cos x_2 A_1(s) + \sin x_1 \cos x_2 A_2(s) + \sin x_2 A_3(s)$. Since p is mapped into the $\gamma(s), \forall s$, $A_1(s)$ is the coordinate representation of the integral curve γ. Moreover, as the matrix belongs to G_2 we have that $A_3(s) = \gamma(s) \times A_2(s)$. Note that it is straightforward to check that any such surface is a CR-surface for which the invariant distribution is totally geodesic.

As the constant vector field V corresponds with e_4, at points of $\gamma(s)$, i.e. at the points where $x_1 = x_2 = 0$ we have $\gamma(s) \times e_4 = -\sigma E_3 + \rho E_1 \times E_3$. Hence, the vector field $W(s) = (\gamma \times e_4) \times \gamma'$ is collinear with $A_2(s)$.

Note that the vector field W can vanish if and only if e_4 is parallel with $\gamma \times \gamma'$. As those points the constant normal e_4 would only have a component in the direction of $J\mathcal{D}_2$. These are exactly the points which we excluded from our open dense subset. Hence this case can not happen.

Therefore, choosing at the initial point the sign of e_4 appropriately we have $A_2 = -W/\|W\|$. Now, at $\gamma(s)$ the vector fields $\gamma, A_2, A_3 = \gamma \times A_2, \gamma', \gamma \times \gamma', A_2 \times \gamma', A_3 \times \gamma'$ form the G_2 basis, and further γ (as well as all of its derivatives), A_2, A_3 are orthogonal to e_4. Moreover, the following holds.

Lemma 5.6. *The integral curve γ satisfies $\langle \gamma'', \gamma \times \gamma' \rangle = \langle \gamma'', \gamma \times e_4 \rangle = 0$.*

Proof. From Theorem 5.3, we have $\gamma'' = D_{E_3} E_3|_{\gamma(s)} = \langle \gamma'', A_2 \rangle A_2 + \langle \gamma'', A_3 \rangle A_3 - \gamma$, so straightforwardly we get that the curve γ has to satisfy

$$\langle \gamma'', \gamma' \rangle = \langle \gamma'', \gamma \times \gamma' \rangle = \langle \gamma'', A_2 \times \gamma' \rangle = \langle \gamma'', A_3 \times \gamma' \rangle = 0. \tag{22}$$

Since, $\|\gamma'\| = 1$ the first condition is trivially satisfied.
Further, as $A_2 || W$ and (3) we get

$$\langle \gamma'', W \times \gamma' \rangle = \langle \gamma'', ((\gamma \times e_4) \times \gamma') \times \gamma' \rangle = \langle \gamma'', \langle (\gamma \times e_4), \gamma' \rangle \gamma' - \gamma \times e_4 \rangle = -\langle \gamma'', \gamma \times e_4 \rangle$$

from which we deduce the second equation. Finally, as $A_3 = \gamma \times A_2$ and

$$\begin{aligned} (\gamma \times W) \times \gamma' &= (\gamma \times ((\gamma \times e_4) \times \gamma')) \times \gamma' = ((-\gamma \times (\gamma \times e_4)) \times \gamma' - \langle \gamma \times e_4, \gamma' \rangle \gamma) \times \gamma' \\ &= (e_4 \times \gamma' - \langle \gamma \times e_4, \gamma' \rangle \gamma) \times \gamma' = -e_4 - \langle \gamma \times e_4, \gamma' \rangle \gamma \times \gamma', \end{aligned}$$

we see that

$$\langle \gamma'', (\gamma \times W) \times \gamma' \rangle = \langle \gamma \times e_4, \gamma' \rangle \langle \gamma \times \gamma', \gamma'' \rangle$$

and therefore $\langle \gamma'', A_3 \times \gamma' \rangle = 0$ does not yield any additional condition. $\square$

LEMMA 5.7. *It holds* $\langle A_2, A_2' \rangle = \langle A_3, A_3' \rangle = \langle A_2, A_3' \rangle = \langle A_2', A_3 \rangle = \langle \gamma, A_2' \rangle = \langle \gamma, A_3' \rangle = 0$. *Moreover the vector field* W *has constant length.*

PROOF. Since $\|A_2\| = \|A_3\| = 1$ we have $\langle A_2, A_2' \rangle = \langle A_3, A_3' \rangle = 0$.
By deriving $\langle A_2, \gamma \rangle = 0$ we get $\langle A_2', \gamma \rangle = -\langle A_2, \gamma' \rangle = -\langle E_1(\gamma), E_3(\gamma) \rangle = 0$, and similarly $\langle A_3', \gamma \rangle = 0$.
Also, by deriving $\langle A_2, A_3 \rangle = 0$ we get $\langle A_2, A_3' \rangle = \langle A_2', A_3 \rangle$.

Further, using (3) and (4) we find that

$$\begin{aligned}
W' &= (\gamma' \times e_4 + 0) \times \gamma' + (\gamma \times e_4) \times \gamma'' = e_4 + (\gamma \times e_4) \times \gamma'', \\
\langle W', W \rangle &= \langle (\gamma \times e_4) \times \gamma'', (\gamma \times e_4) \times \gamma' \rangle = -\langle (\gamma \times e_4) \times ((\gamma \times e_4) \times \gamma'), \gamma'' \rangle \\
&= -\langle \langle \gamma \times e_4, \gamma' \rangle \gamma \times e_4 - \langle \gamma \times e_4, \gamma \times e_4 \rangle \gamma', \gamma'' \rangle = -\langle \gamma \times e_4, \gamma' \rangle \langle \gamma \times e_4, \gamma'' \rangle = 0.
\end{aligned}$$

This last equation immediately implies that W has constant length and we get moreover that

$$\begin{aligned}
\langle A_2', A_3 \rangle &= -\frac{1}{\|W\|^2} \langle W', \gamma \times W \rangle = -\frac{1}{\|W\|^2} \langle e_4 + (\gamma \times e_4) \times \gamma'', e_4 \times \gamma' - \langle \gamma \times e_4, \gamma' \rangle \gamma \rangle \\
&= -\frac{1}{\|W\|^2} (\langle (\gamma \times e_4) \times \gamma'', e_4 \times \gamma' \rangle - \langle \gamma \times e_4, \gamma' \rangle \langle (\gamma \times e_4) \times \gamma'', \gamma \rangle) \\
&= -\frac{1}{\|W\|^2} (-\langle (e_4 \times \gamma) \times \gamma'', e_4 \times \gamma' \rangle - \langle \gamma \times e_4, \gamma' \rangle \langle -\gamma \times (e_4 \times \gamma'') + 2\langle \gamma, \gamma'' \rangle e_4, \gamma \rangle) \\
&= \frac{1}{\|W\|^2} \langle (e_4 \times \gamma) \times \gamma'', e_4 \times \gamma' \rangle - 0 = \frac{1}{\|W\|^2} \langle -e_4 \times (\gamma \times \gamma'') - \langle \gamma, \gamma'' \rangle e_4, e_4 \times \gamma' \rangle \\
&= \frac{1}{\|W\|^2} \langle -e_4 \times (\gamma \times \gamma''), e_4 \times \gamma' \rangle = -\frac{1}{\|W\|^2} \langle \gamma \times \gamma'', \gamma' \rangle = 0.
\end{aligned}$$

$\square$

We know compute the tangent space to the immersion F. It is spanned by

$$\begin{aligned}
F_{x_1} &= -\cos x_2 \sin x_1 \gamma + \cos x_2 \cos x_1 A_2, \\
F_{x_2} &= -\sin x_2 \cos x_1 \gamma - \sin x_2 \sin x_1 A_2 + \cos x_2 A_3, \\
F_s &= \cos x_1 \cos x_2 \gamma' + \cos x_2 \sin x_1 A_2' + \sin x_2 A_3',
\end{aligned}$$

and clearly, $\gamma \times \dfrac{F_{x_1}}{\cos x_2} = F_{x_2}$ so $\mathcal{D}_1 = Span(F_{x_1}, F_{x_2})$. Since $\langle F_s, F_{x_1} \rangle = \langle F_s, F_{x_2} \rangle = 0$ it follows $\mathcal{D}_2 = Span(F_s)$. Note $\|\dfrac{F_{x_1}}{\cos x_2}\| = \|F_{x_2}\| = 1$. Moreover, straightforward computation shows that

$$\begin{aligned}
F_{x_1 x_1} &= \nabla_{\partial x_1} \partial x_1 + h(\partial x_1, \partial x_1) - \langle \partial x_1, \partial x_1 \rangle F = \sin x_2 \cos x_2 F_{x_2} - \cos^2 x_2 F, \\
F_{x_2 x_2} &= \nabla_{\partial x_2} \partial x_2 + h(\partial x_2, \partial x_2) - \langle \partial x_2, \partial x_2 \rangle F = -F, \\
F_{x_1 x_2} &= \nabla_{\partial x_1} \partial x_2 + h(\partial x_1, \partial x_2) - \langle \partial x_1, \partial x_2 \rangle F = -\tan x_2 F_{x_1},
\end{aligned}$$

so $h(\partial x_1, \partial x_1) = h(\partial x_1, \partial x_2) = h(\partial x_2, \partial x_2) = 0$, or $h(\mathcal{D}_1, \mathcal{D}_1) = 0$.

Now,

$$F_{ss} = \cos x_1 \cos x_2 \gamma'' + \cos x_2 \sin x_1 A_2'' + \sin x_2 A_3'' = \nabla_{\partial s}\partial s + h(\partial s, \partial s) - \langle \partial s, \partial s\rangle F,$$

and if we denote by $B = \langle F_{ss}, \frac{F_{x_1}}{\cos x_2}\rangle \frac{F_{x_1}}{\cos x_2} + \langle F_{ss}, F_{x_2}\rangle F_{x_2} + \langle F_{ss}, F\rangle F$ we conclude that $h(\mathcal{D}_2, \mathcal{D}_2)$ vanishes, or equivalently that M is minimal, and moreover $h(\mathcal{D}_2, \mathcal{D}_2) = 0$ if and only if $F_{ss} - B$ is collinear to F_s. A straightforward computation shows

$$\begin{aligned} F_{ss} - B = {} & \cos x_1 \cos x_2[\gamma'' - \langle \gamma'', A_2\rangle A_2 - \langle \gamma'', A_3\rangle A_3 + \gamma] \\ & + \sin x_1 \cos x_2 [A_2'' - \langle A_2'', A_2\rangle A_2 - \langle A_2'', A_3\rangle A_3 - \langle A_2'', \gamma\rangle \gamma] \\ & + \sin x_2 [A_3'' - \langle A_3'', A_2\rangle A_2 - \langle A_3'', A_3\rangle A_3'' - \langle A_3, \gamma\rangle \gamma]. \end{aligned}$$

As the vector fields $\gamma, A_2, A_3 = \gamma \times A_2, \gamma', \gamma \times \gamma', A_2 \times \gamma', A_3 \times \gamma'$ form a G_2 basis, we see from (22) that γ'' lies in the space spanned by A_2, A_3 and γ. Therefore

$$\begin{aligned} F_{ss} - B = {} & \sin x_1 \cos x_2 [A_2'' - \langle A_2'', A_2\rangle A_2 - \langle A_2'', A_3\rangle A_3 - \langle A_2'', \gamma\rangle \gamma] \\ & + \sin x_2 [A_3'' - \langle A_3'', A_2\rangle A_2 - \langle A_3'', A_3\rangle A_3'' - \langle A_3, \gamma\rangle \gamma]. \end{aligned}$$

If we denote the projections of A_2'' and A_3'' on $Span(\gamma', \gamma\times\gamma', A_2\times\gamma', A_3\times\gamma')$ by $A_2''^{\perp}$ and $A_3''^{\perp}$ we have $F_{ss} - B = \sin x_1 \cos x_2 A_2''^{\perp} + \sin x_2 A_3''^{\perp}$. $F_{ss} - B$ has to be collinear to F_s at all points, in particular for $x_1 \neq 0, x_2 = 0$, or $x_1 = 0, x_2 \neq 0$ which implies

$$A_2''^{\perp} \;||\; \cos x_1 \gamma' + \sin x_1 A_2', \quad A_3''^{\perp} \;||\; \cos x_2 \gamma' + \sin x_2 A_3'.$$

If we, for instance, assume $A_2''^{\perp} \neq 0$, since we can choose arbitrary x_2, it follows A_2' is collinear to γ'. A similar property holds also for $A_3''^{\perp}$. We now consider 4 subcases.

Case 1: $A_2''^{\perp} \neq 0 \neq A_3''^{\perp}$. As the length of W is constant this implies that both W' and $(\gamma \times W)' = \gamma' \times W + \gamma \times W'$ are parallel to γ'. We can write $W' = f_1 \gamma'$. Substituting this in the second equation we have that

$$(\gamma \times W)' = \gamma' \times (W - f_1\gamma) \parallel \gamma'.$$

This can only happen if we can write

$$W = f_1 \gamma + f_2 \gamma'.$$

Given the definition of A_2, this contradicts that the vector fields $\gamma, A_2, A_3 = \gamma \times A_2, \gamma', \gamma\times\gamma', A_2\times\gamma', A_3\times\gamma'$ form a G_2 basis. Therefore this case can not occur.

Case 2: $A_2''^{\perp} \neq 0$ and $A_3''^{\perp} = 0$. In this case we know that A_2' is parallel with γ' but that $A_2 \notin Span(\gamma, A_2, A_3)$. Note that from the multiplication table it follows that $Span(\gamma, A_2, A_3)$ is closed under multiplication. Moreover, we have that

$$\begin{aligned}
A_3 &= \gamma \times A_2 \\
A_3' &= \gamma' \times A_2 + \gamma \times A_2' \\
A_3'' &= \gamma'' \times A_2 + 2\gamma' \times A_2 + \gamma \times A_2'' = \gamma'' \times A_2 + \gamma \times A_2''.
\end{aligned}$$

This implies that $\gamma \times A_2'' \in Span(\gamma, A_2, A_3)$. Multiplying oncemore with γ, it follows from (3) that also $A_2'' \in Span(\gamma, A_2, A_3)$ which is a contradiction.
Case 3: $A_2''^{\perp} = 0$ and $A_3''^{\perp} \neq 0$. A contradiction follows in a similar way as in the previous case.
Case 4: $A_2''^{\perp} = 0 = A_3''^{\perp}$. As all the previous cases let to a contradiction, this is the only possibility. In that case we get the following lemma:

LEMMA 5.8. *The integral curve γ satisfies*

$$\langle (\gamma \times e_4) \times \gamma''', \gamma' \rangle = \langle e_4 \times \gamma''', \gamma' \rangle = 0.$$

PROOF. We know that

$$A_2'', A_3'' \in Span(\gamma, A_2, A_3). \tag{23}$$

Note that

$$A_3'' = \gamma'' \times A_2 + 2\gamma' \times A_2' + \gamma \times A_2'',$$

and since $\gamma'' \in Span(\gamma, A_2, A_3)$, the (23) is equivalent to $\gamma' \times A_2', A_2'' \in Span(\gamma, A_2, A_3)$. However, since $A_2' || W'$ and using (4) we see that

$$\begin{aligned}
&\langle \gamma' \times W', \gamma' \rangle = 0, && \langle \gamma' \times W', \gamma' \times \gamma \rangle = \langle \gamma, W' \rangle = 0, \\
&\langle \gamma' \times W', \gamma' \times A_2 \rangle = \langle A_2, W' \rangle = 0, && \langle \gamma' \times W', \gamma' \times A_3 \rangle = \langle A_3, W' \rangle = 0,
\end{aligned}$$

and therefore $\gamma' \times A_2' \in Span(\gamma, A_2, A_3)$ does not give any new conditions.

We have

$$W'' = (\gamma' \times e_4) \times \gamma'' + (\gamma \times e_4) \times \gamma'''.$$

The condition that $A_2'' \in Span(\gamma, A_2, A_3)$ gives the following.

$$0 = \langle W'', \gamma' \rangle = \langle -\gamma' \times (e_4 \times \gamma'') + 2\langle \gamma, \gamma'' \rangle e_4, \gamma' \rangle + \langle (\gamma \times e_4) \times \gamma''', \gamma' \rangle = \langle (\gamma \times e_4) \times \gamma''', \gamma' \rangle,$$

By deriving $\langle \gamma, \gamma'' \rangle = -1$ we get $\langle \gamma, \gamma''' \rangle = 0$ and then

$$\begin{aligned}
0 = \langle W'', \gamma \times \gamma' \rangle &= -\langle \gamma' \times (e_4 \times \gamma''), \gamma \times \gamma' \rangle + \langle -\gamma \times (e_4 \times \gamma''') + \langle \gamma, \gamma''' \rangle e_4, \gamma \times \gamma' \rangle \\
&= \langle e_4 \times \gamma'', \gamma \rangle - \langle e_4 \times \gamma''', \gamma' \rangle + \langle \gamma, \gamma''' \rangle \langle \gamma \times \gamma', e_4 \rangle = -\langle e_4 \times \gamma''', \gamma' \rangle.
\end{aligned}$$

Further,

$$
\begin{aligned}
0 &= \langle W'', A_2 \times \gamma' \rangle = \langle \gamma' \times (e_4 \times \gamma''), \gamma' \times A_2 \rangle + \langle (\gamma \times e_4) \times \gamma''', A_2 \times \gamma' \rangle \\
&= \langle A_2, e_4 \times \gamma'' \rangle + \langle (\gamma \times e_4) \times \gamma''', A_2 \times \gamma' \rangle \\
&= \frac{1}{\|W\|} (\langle (e_4 \times \gamma) \times \gamma', e_4 \times \gamma'' \rangle - \langle (\gamma \times e_4) \times \gamma''', ((\gamma \times e_4) \times \gamma') \times \gamma' \rangle) \\
&= -\frac{1}{\|W\|} (\langle e_4 \times (\gamma \times \gamma'), e_4 \times \gamma'' \rangle + \langle (\gamma \times e_4) \times \gamma''', \langle \gamma', \gamma \times e_4 \rangle \gamma' - \gamma \times e_4 \rangle) \\
&= \frac{1}{\|W\|} (-\langle \gamma'', \gamma \times \gamma' \rangle - \langle \gamma', \gamma \times e_4 \rangle \langle (\gamma \times e_4) \times \gamma''', \gamma' \rangle) \\
&= -\frac{1}{\|W\|} \langle \gamma', \gamma \times e_4 \rangle \langle (\gamma \times e_4) \times \gamma''', \gamma' \rangle,
\end{aligned}
$$

and,

$$
\begin{aligned}
0 &= \langle W'', A_3 \times \gamma' \rangle = \frac{1}{\|W\|} \langle (\gamma' \times e_4) \times \gamma'' + (\gamma \times e_4) \times \gamma''', e_4 + \langle \gamma \times e_4, \gamma' \rangle \gamma \times \gamma' \rangle \\
&= \frac{1}{\|W\|} \langle e_4 \times (\gamma' \times \gamma'') + e_4 \times (\gamma \times \gamma'''), e_4 + \langle \gamma \times e_4, \gamma' \rangle \gamma \times \gamma' \rangle \\
&= \frac{\langle \gamma \times e_4, \gamma' \rangle}{\|W\|} (\langle (e_4 \times \gamma'') \times \gamma', \gamma \times \gamma' \rangle + \langle (e_4 \times \gamma''') \times \gamma, \gamma \times \gamma' \rangle) \\
&= \frac{\langle \gamma \times e_4, \gamma' \rangle}{\|W\|} (\langle \gamma, e_4 \times \gamma'' \rangle - \langle \gamma', e_4 \times \gamma''' \rangle),
\end{aligned}
$$

does not give any additional information. □

Therefore we have:

THEOREM 5.9. *Let M be a minimal three-dimensional CR submanifold of $S^6(1)$ which is not linearly full in $S^6(1)$. Then M is locally congruent to the immersion*

$$F(x_1, x_2, s) = \cos x_1 \cos x_2 \gamma(s) + \sin x_1 \cos x_2 A_2(s) + \sin x_2 A_3(s) \tag{24}$$

where γ is a sphere curve that satisfies the following

$$
\begin{aligned}
&\gamma \perp e_4, \quad \|\gamma'\| = 1, \quad \langle \gamma'', \gamma \times \gamma' \rangle = \langle \gamma'', \gamma \times e_4 \rangle = 0, \\
&\langle (\gamma \times e_4) \times \gamma''', \gamma' \rangle = \langle e_4 \times \gamma''', \gamma' \rangle = 0,
\end{aligned} \tag{25}
$$

and $A_2 = -\dfrac{(\gamma \times e_4) \times \gamma'}{\|(\gamma \times e_4) \times \gamma'\|}$ and $A_3 = \gamma \times A_2$. Conversely, if γ is a sphere curve that satisfies conditions (25), then (24) is a minimal CR immersion into the sphere S^6 which is not linearly full.

PROOF. We have already seen that on an open dense subset M can be written as above. Also as at each stage we verified that there are no additional conditions, a the straightforward computation shows that for a sphere curve γ that satisfies (25), the immersion (24) satisfies the conditions of the theorem. □

REMARK 5.10. We will now see how the examples of Hashimoto and Mashimo can be interpreted in the above framework. We write

$$F_{\lambda_1\lambda_2}((y_1,y_2,y_3),s) = y_1(\cos(\lambda_1 s)e_1 + \sin(\lambda_1 s)e_5) \\ + y_2(\cos(\lambda_2 s)e_2 + \sin(\lambda_2 s)e_5) + y_3(\cos((\lambda_1+\lambda_2)s)e_3 - \sin((\lambda_1+\lambda_2)s)e_7),$$

where $y_1^2+y_2^2+y_3^2=1$ and $\{e_1,\dots,e_7\}$ is a G_2-frame.

Note that replacing (λ_1,λ_2) by a multiple of itself yields the same CR-submanifold. Also it is easy to check that we can not apply the trivial choice. Namely if we take $y_1 = \cos x_1 \cos x_2$, $y_2 = \sin x_1 \cos x_2$ and $y_3 = \sin x_2$ and work in the neighborhood of the point $(0,0,0)$, it follows that

$$\langle e_4, F_s(s,0,0)\times F_{x_1}(s,0,0)\rangle = \langle e_4, F_s(s,0,0)\times F_{x_2}(s,0,0)\rangle = 0,$$

and therefore the point $(0,0,0)$ does not belong to the open dense subset on which we worked during the proof.

In order to overcome this problem and still be able to work at the point $(0,0,0)$, we take a different parametrization of the sphere. We take angles a and b (at the moment arbitrary) and define

$$\begin{aligned} y_1 &= \cos(a)\cos(b)\cos(x_2)\cos(x_1) - \cos(a)\sin(b)\sin(x_2) - \sin(a)\sin(x_1)\cos(x_2),\\ y_2 &= \sin(a)\cos(b)\cos(x_1)\cos(x_2) - \sin(a)\sin(b)\sin(x_2) + \cos(a)\sin(x_1)\cos(x_2),\\ y_3 &= \sin(b)\cos(x_1)\cos(x_2) + \cos(b)\sin(x_2). \end{aligned}$$

An elementary computation shows that

$$\begin{aligned} \langle F_{x_1}, F_{x_1}\rangle &= \cos^2 x_2, & \langle F_{x_2}, F_{x_2}\rangle &= 1,\\ \langle F_{x_1}, F_{x_2}\rangle &= 0, & J\tfrac{F_{x_1}}{\cos x_2} &= F_{x_2}. \end{aligned}$$

We put $E_1 = \frac{F_{x_1}}{\cos x_2}$ and $E_2 = F_{x_2}$. A straightforward computation shows that

$$\langle F_s, E_1\rangle = \langle F_s, E_2\rangle = 0.$$

As

$$\langle F_s(s,0,0), F_s(s,0,0)\rangle = \cos^2(b)\left({\lambda_1}^2\cos^2(a) + {\lambda_2}^2\sin^2(a)\right) + \sin^2(b)(\lambda_1+\lambda_2)^2$$

we see that we can rescale (λ_1,λ_2) such that $\langle F_s(s,0,0), F_s(s,0,0)\rangle = 1$ and therefore $F_s(s,0,0)$ is the integral curve of E_3 through $(0,0,0)$.

A straightforward computation also shows that

$$\begin{aligned} \langle e_4, F_s(s,0,0)\times F_{x_1}(s,0,0)\rangle &= \sin(a)\cos(a)\cos(b)(\lambda_2-\lambda_1),\\ \langle e_4, F_s(s,0,0)\times F_{x_2}(s,0,0)\rangle &= -\frac{1}{4}\sin(2b)(\cos(2a)(\lambda_1-\lambda_2) + 3(\lambda_1+\lambda_2)),\\ \langle e_4, JF_s(s,0,0)\rangle &= \frac{1}{4}\left(2\cos(2a)\cos^2(b)(\lambda_2-\lambda_1) - 3\cos(2b)(\lambda_1+\lambda_2) + (\lambda_1+\lambda_2)\right). \end{aligned}$$

So we see that, in order to be consistent with our proof we have to choose a and b in such a way that the first expression vanishes and the other two are not zero. For example

1. if $2\lambda_1+\lambda_2\neq 0$ and $\lambda_2\neq 0$, we can take $a=0$ and $b=\frac{\pi}{4}$,
2. if $2\lambda_2+\lambda_1\neq 0$ and $\lambda_1\neq 0$, we can take $a=\frac{\pi}{2}$ and $b=\frac{\pi}{4}$.

Note that the above two cases cover all possibilities.
Case 1: $a=0$ and $b=\frac{\pi}{4}$. We find

$$\gamma(s)=\left(\frac{\cos(s\lambda_1)}{\sqrt{2}},0,\frac{\cos(s(\lambda_1+\lambda_2))}{\sqrt{2}},0,\frac{\sin(s\lambda_1)}{\sqrt{2}},0,-\frac{\sin(s(\lambda_1+\lambda_2))}{\sqrt{2}}\right).$$

Moreover we also have that

$$W(s)=\left(0,-\frac{1}{2}(2\lambda_1+\lambda_2)\cos(s\lambda_2),0,0,0,-\frac{1}{2}(2\lambda_1+\lambda_2)\sin(s\lambda_2),0\right),$$

which determines (upto a choice of sign) the vector field $A_2(s)$. However, if we take

$$A_2(s)=(0,\cos(s\lambda_2),0,0,0,\sin(s\lambda_2),0)\,,$$

and

$$A_3(s)=\gamma(s)\times A_2(s)=(-\frac{\cos(\lambda_1 s)}{\sqrt{2}},0,\frac{\cos((\lambda_1+\lambda_2)s)}{\sqrt{2}},0,-\frac{\sin(\lambda_1 s)}{\sqrt{2}},0,-\frac{\sin((\lambda_1+\lambda_2)s)}{\sqrt{2}})$$

we deduce indeed that

$$F(s,x_1,x_2)=\gamma(s)\cos x_1\cos x_2+A_2(s)\sin x_1\cos x_2+A_3(s)\sin x_2.$$

Case 2: $a=0$ and $b=\frac{\pi}{4}$. We find

$$\gamma(s)=\left(0,\frac{\cos(s\lambda_2)}{\sqrt{2}},\frac{\cos(s(\lambda_1+\lambda_2))}{\sqrt{2}},0,0,\frac{\sin(s\lambda_2)}{\sqrt{2}},-\frac{\sin(s(\lambda_1+\lambda_2))}{\sqrt{2}}\right)$$

Moreover we also have that

$$W(s)=\left(\frac{1}{2}(\lambda_1+2\lambda_2)\cos(s\lambda_1),0,0,0,\frac{1}{2}(\lambda_1+2\lambda_2)\sin(s\lambda_1),0,0\right),$$

which determines (upto a choice of sign) the vector field $A_2(s)$. However, if we take

$$A_2(s)=(-\cos(s\lambda_1),0,0,0,-\sin(s\lambda_1),0,0)\,,$$

and therefore

$$A_3(s)=\gamma(s)\times A_2(s)=\left(0,-\frac{\cos(s\lambda_2)}{\sqrt{2}},\frac{\cos(s(\lambda_1+\lambda_2))}{\sqrt{2}},0,0,-\frac{\sin(s\lambda_2)}{\sqrt{2}},-\frac{\sin(s(\lambda_1+\lambda_2))}{\sqrt{2}}\right)$$

we deduce indeed that

$$F(s,x_1,x_2)=\gamma(s)\cos x_1\cos x_2+A_2(s)\sin x_1\cos x_2+A_3(s)\sin x_2.$$

Note that the above procedure already indicates that the same submanifold can be written in many ways, using different curves γ, satisfying the conditions of our theorem. Therefore in order to get a more explicit result we are now going to determine the curves γ more explicitly (and try to give a less complicated expression for the immersion). We recall that the frame $\gamma, A_2, A_3 = \gamma \times A_2, \gamma', \gamma \times \gamma', A_2 \times \gamma', A_3 \times \gamma'$ form a G_2 basis. Moreover, we have that $W = (\gamma \times e_4) \times \gamma'$, from which we deduce that $\|W\|^2 = 1 - \langle \gamma \times \gamma', e_4 \rangle^2$. Therefore, from the proof of the theorem and the choice of E_1 it follows that we can write

$$e_4 = \cos\theta \gamma \times \gamma' + \sin\theta A_3 \times \gamma'.$$

Moreover Lemma 5.4 implies that θ is a constant. From (22) we see that we can write

$$\gamma'' = -\gamma + \kappa_1 A_2 + \kappa_2 A_3.$$

It follows now from Lemma 5.5 and Lemma 5.6 that κ_1 and κ_2 are both constants. Therefore we can write down a system of ordinary differential equations with constant coefficients for the derivatives of our frame. Writing $f_1 = \gamma, f_2 = A_2, f_3 = A_3 = \gamma \times A_2, f_4 = \gamma', f_5 = \gamma \times \gamma', f_6 = A_2 \times \gamma', f_7 = A_3 \times \gamma'$, and $F = (f_1 f_2 f_3 f_4 f_5 f_6 f_7)$, we get that

$$F' = F \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_1 & -\kappa_2 & \kappa_1 \cot(\theta) & \kappa_2 \cot(\theta) \\ 0 & 0 & 0 & \kappa_2 & \kappa_1 & \kappa_2 \cot(\theta) + 1 & -\kappa_1 \cot(\theta) \\ 1 & -\kappa_1 & -\kappa_2 & 0 & 0 & 0 & 0 \\ 0 & \kappa_2 & -\kappa_1 & 0 & 0 & 0 & 0 \\ 0 & -\kappa_1 \cot(\theta) & -\kappa_2 \cot(\theta) - 1 & 0 & 0 & 0 & 0 \\ 0 & -\kappa_2 \cot(\theta) & \kappa_1 \cot(\theta) & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, we have

$$\begin{aligned} A_2'' &= \kappa_1 \gamma - \csc^2\theta(\kappa_1^2 + \kappa_2^2) A_2 - \kappa_1 \cot\theta A_3, \\ A_3'' &= \kappa_1 \gamma - \kappa_1 \cot\theta A_2 - (\csc^2\theta(\kappa_1^2 + \kappa_2^2) + 2\kappa_2 \cot\theta + 1) A_3. \end{aligned} \tag{26}$$

Let μ_1, μ_2, μ_3 be the eigenvalues of the symmetric matrix

$$A = \begin{pmatrix} -\cos\theta & 0 & -\sin\theta \\ 0 & -\kappa_2 \csc\theta & \kappa_1 \csc\theta \\ -\sin\theta & \kappa_1 \csc\theta & \kappa_2 \csc\theta + \cos\theta \end{pmatrix},$$

and $(a_{1i}, a_{2i}, a_{3i}), i = 1, 2, 3$ the corresponding scaled eigenvectors, so that $(a_{ij}) \in SO(3)$. An eigenvalue can be zero, only if the $\det A = \csc\theta(\cot\theta(\kappa_1^2 + \kappa_2^2) + \kappa_2) = 0$. Assume that it is the case. Then at the points $\gamma(s)$ the first normal space, which is spanned by the normal parts of F_{1s} and F_{2s}, i.e. by the normal parts of A_2' and A_3', which are respectively, $\kappa_2 f_5 - \kappa_1 \cot\theta f_6 - \kappa_2 \cot\theta f_7$ and $-\kappa_1 f_5 - (\kappa_2 \cot\theta + 1) f_6 + \kappa_1 \cot\theta f_7$, is one-dimensional, since they are collinear. That means that along γ the function x of the Example 5.4 is zero. Since also $E_1(x) = 0$ we easily obtain that then x vanishes, and submanifold satisfies the Chens equality. Therefore, from now on we may assume that non of the eigenvalues of the matrix A is zero.

Note that $TraceA = 0$ so $\mu_1+\mu_2+\mu_3 = 0$. Then, $(a_{1i}, a_{2i}, a_{3i}), i = 1, 2, 3$ are eigenvectors for the matrix $-A^2$ and for the eigenvalues, respectively, $-\mu_i^2, i = 1, 2, 3$. Straightforward computation and (26) show that this further implies

$$a_{1i}\gamma'' + a_{2i}A_2'' + a_{3i}A_3'' = -\mu_i^2(a_{1i}\gamma' + a_{2i}A_2 + a_{3i}A_3), i = 1, 2, 3$$

and further

$$a_{1i}\gamma + a_{2i}A_2 + a_{3i}A_3 = C_i \cos(\mu_i s) + D_i \sin(\mu_i s), i = 1, 2, 3, \tag{27}$$

where $C_i, D_i, i = 1, 2, 3$ are constant vectors. Since the rows of matrix (a_{ij}) are unit it follows $\langle C_i, C_i\rangle = \langle D_i, D_i\rangle = 1$ and $\langle C_i, D_i\rangle = 0$, and the orthogonality of the rows implies $\langle C_i, C_j\rangle = \langle D_i, D_j\rangle = \langle C_i, D_j\rangle, i \neq j$.

Note also that we have picked initial conditions such that $\gamma(0) = e_1$, $A_2(0) = e_2$ and $A_3(0) = e_3$ and $e_4 = (\cos\theta e_1 + \sin\theta e_3) \times \gamma'(0)$. This implies that

$$\gamma'(0) = -(\cos\theta e_1 + \sin\theta e_3) \times e_4 = -\cos\theta e_5 - \sin\theta e_7,$$

end further

$$\begin{aligned} A_2'(0) &= -\kappa_2 \csc\theta e_6 + \kappa_1 \csc\theta e_7, \\ A_3'(0) &= -\sin\theta e_5 + \kappa_1 \csc\theta e_6 + (\kappa_2 \csc\theta + \cos\theta)e_7. \end{aligned} \tag{28}$$

Now,(27) and its derivative for $s = 0$ imply that $C_i = a_{1i}e_1 + a_{2i}e_2 + a_{3i}e_3$ and $\mu_i D_i = a_{1i}(-\cos\theta e_5 - \sin\theta e_7) + a_{2i}(-\kappa_2 \csc\theta e_6 + \kappa_1 \csc\theta e_7) + a_{3i}(-\sin\theta e_5 + \kappa_1 \csc\theta e_6 + (\kappa_2 \csc\theta + \cos\theta)e_7)$. But, (a_{1i}, a_{2i}, a_{3i}) is an eigenvector for matrix A and eigenvalue μ_i so along with (28) this implies $D_i = -e_4 \times C_i = a_{1i}e_5 + a_{2i}e_6 + a_{3i}e_7$.

Matrix A defines a reparametrization of the sphere $S^2 : y_1^2+y_2^2+y_3^2 = 1$ given by $(y_1y_2y_3)^t = A(z_1z_2z_3)^t$, and also defines an isometry which maps e_1, e_2, e_3 respectively into $C_1 = \overline{e}_1, C_2 = \overline{e}_2, C_3 = \overline{e}_2$ that along with e_4, $D_1 = \overline{e}_5, D_2 = \overline{e}_6, D_3 = \overline{e}_7$ form a G_2 basis. This transforms an immersion $F(s, y_1, y_2, y_3) = y_1\gamma(s) + y_2A_2(s) + y_3A_3(s)$ into

$$\begin{aligned} \overline{F}(s, z_1, z_2, z_3) = {} & (\cos(\mu_1)\overline{e}_1 + \sin(\mu_1)\overline{e}_5)z_1 + (\cos(\mu_2)\overline{e}_2 + \sin(\mu_2)\overline{e}_6)z_2 \\ & + (\cos(\mu_3)\overline{e}_3 + \sin(\mu_3)\overline{e}_7)z_3, \quad \mu_1 + \mu_2 + \mu_3 = 0. \end{aligned}$$

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